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Fuchsian Triangle Groups and Grothendieck Dessins. Variations on a Theme of Belyi

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Abstract: According to a theorem of Belyi, a smooth projective algebraic curve is defined over a number field if and only if there exists a non-constant element of its function field ramified only over 0, 1 and ∞. The existence of such a Belyi function is equivalent to that of a representation of the curve as a possibly compactified quotient space of the Poincaré upper half plane by a subgroup of finite index in a Fuchsian triangle group. On the other hand, Fuchsian triangle groups arise in many contexts, such as in the theory of hypergeometric functions and certain triangular billiard problems, which would appear at first sight to have no relation to the Galois problems that motivated the above discovery of Belyi. In this note we review several results related to Belyi's theorem and we develop certain aspects giving examples. For preliminary accounts, see the preprint [Wo1], the conference proceedings article [BauItz] and the "Comptes Rendus" note [CoWo2].

0. Introduction

While several years ago the moduli space of compact Riemann surfaces seemed to be as remote a subject of consideration in theoretical physics as possible, it plays nowadays a considerable role from various points of view. To name some: string field theory, conformal statistical physics, topological field theories, two dimensional quantum gravity, classical integrable systems. On the other hand it is a classical subject in mathematics since the days of Riemann involving intricate structures. Any further one added to the impressive amount unraveled so far is nevertheless welcome to get a deeper understanding. According to a well established tradition, a natural temptation for physicists was to develop a manageable discretization. This was accomplished through the use of matrix models and their perturbative expansions in terms of decorated cell decompositions of surfaces, leading to a corresponding cellular complex in moduli space. This opened the way to a fascinating interplay between topological properties of moduli spaces and (a restricted class of) solutions