Why the General Zakharov–Shabat Equations Form a Hierarchy

L.A. Dickey

Dept. Math., University of Oklahoma, Norman, OK 73019, USA. Internet:
ldickey@nsfuvax.math.uoknor.edu

Abstract. The totality of all Zakharov–Shabat equations (ZS), i.e., the zero-curvature
equations with rational dependence on a spectral parameter, if properly defined, can
be considered as a hierarchy. The latter means a collection of commuting vector
fields in the same phase space. Further properties of the hierarchy are discussed, such
as additional symmetries, an analogue to the string equation, a Grassmannian
related to the ZS hierarchy, and a Grassmannian definition of soliton solutions.

0. Introduction.

We are accustomed to the fact that integrable systems appear not one at a time but
in big families called hierarchies. So, first of all, the KdV ($n = 2$) hierarchy was
invented (Gardner, Green, Kruskal, and Miura made the first and the most
important discovery, the KdV equation in the proper sense; later on all the higher
KdV were found by Gardner). Then this was generalized to every n (Gelfand and
Dickey, who used fractional powers of operators). Thus, infinitely many general-
ized KdV hierarchies were found. They were unified to a single one large KP
hierarchy (Kyoto school: Sato et al. [1]). Another line of developments was
connected with equations generated by a linear first order differential operator
with matrix coefficients linearly dependent on a spectral parameter (Albowitz,
Kaup, Newell and Segur for matrices 2×2 and Dubrovin in a general case, let us
call this hierarchy AKNS-D). Later this was generalized to operators with a poly-
nomial dependence on the spectral parameter. Thus, for every degree of a poly-
nomial, m, there is a hierarchy, a generalized AKNS-D.[1]

More than that, there is a very general type of equations proposed by Zakharov
and Shabat (see [2]): the equation of zero curvature, where matrices depend on
some spectral parameter as rational functions. (The above mentioned hierarchies
where operators depend on a parameter as polynomials, i.e., have a single pole at
infinity, represent special case of these equations). The ZS equations usually have
been treated individually, not as a hierarchy.

---

[1] More detail can be found, e.g., in [3].