Commun. Math. Phys. 163, 491–508 (1994)

Communications in
Mathematical
Physics

Convergence of General Decompositions of Exponential Operators

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Received: 14 May 1993/in revised form: 2 September 1993

Abstract: A general theorem is proved concerning the convergence of decompositions of exponential operators in a Banach space (or normed space). As a corollary, the convergence of fractal decompositions is proved. The convergence of generalized Trotter-like formulas is also shown to result from the general theorem.

1. Introduction

In the present paper, we investigate the convergence of some systematic series of decompositions of exponential operators [1 ∼ 5] such as $\exp[x(A_1+A_2+\cdots+A_q)]$ for non-commutable operators $\{A_j\}$ in a Banach space. In this note we mean an operator by a bounded linear operator on a Banach space.

As is well known [1 ∼ 11], the first-order decomposition $Q(x)$ is given by

$$Q(x)=\mathrm{e}^{xA_1}\mathrm{e}^{xA_2}\ldots\mathrm{e}^{xA_{q-1}}\mathrm{e}^{xA_q}\,, \tag{1.1}$$

i.e.,

$$\mathrm{e}^{x(A_1+A_2+\cdots+A_q)}=Q(x)+\mathrm{O}(x^2)\,, \tag{1.2}$$

and the second-order symmetric decomposition is given by

$$S(x)=\mathrm{e}^{\frac{x}{2}A_1}\ldots\mathrm{e}^{\frac{x}{2}A_{q-1}}\mathrm{e}^{xA_q}\mathrm{e}^{\frac{x}{2}A_{q-1}}\ldots\mathrm{e}^{\frac{x}{2}A_1}\,, \tag{1.3}$$

i.e.,

$$\mathrm{e}^{x(A_1+\cdots+A_q)}=S(x)+\mathrm{O}(x^3)\,. \tag{1.4}$$

The above symmetry is characterized by the relation

$$S(x)S(-x)=\mathbb{1}\ \text{or}\ S(-x)=S^{-1}(x)\,. \tag{1.5}$$

In general, the m^{th} order exponential decomposition $Q_m(x)$ is given in the form [1 ∼ 5]

$$Q_m(x)=\mathrm{e}^{x\tau_{11}A_1}\mathrm{e}^{x\tau_{12}A_2}\ldots\mathrm{e}^{x\tau_{1q}A_q}\mathrm{e}^{x\tau_{21}A_1}\mathrm{e}^{x\tau_{22}A_2}\ldots\mathrm{e}^{x\tau_{2q}A_q}\ldots\,, \tag{1.6}$$