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Modules over $\mathfrak{U}_q(\mathfrak{sl}_2)$

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Abstract. The restricted quantum universal enveloping algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ decomposes in a canonical way into a direct sum of indecomposable left (or right) ideals. They are useful for determining the direct summands which occur in the tensor product of two simple $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules. The indecomposable finite-dimensional $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules are classified and located in the Auslander-Reiten quiver.

1. Introduction

One of the basic problems in the theory of quantum universal enveloping algebras is to decompose a tensor product of simple modules into a direct sum of indecomposable ones and hence to elucidate the structure of the corresponding fusion rule algebra. Although this problem is solved for $\mathfrak{U}_q(\mathfrak{sl}_2)$, it might still be interesting to derive the solution in a new way; at least in principle, the method used here can be generalised to higher rank quantum universal enveloping algebras. A distinguishing feature is that neither the quantum Casimir operator nor the R-matrix appears explicitly, nor occurs any tedious calculation whatever. Then, the finite-dimensional $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules are classified, partly because there seems to be some interest in that (see [Sm]). Still, at least the *result* should be known to the experts and also to some readers of [RT].

In Sect. 2 we set forth the algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ at $q = \exp(\pi i m/N)$.

The main issue of Sect. 3 is Theorem 3.7, which states how $\mathfrak{U}_q(\mathfrak{sl}_2)$ decomposes into a direct sum of indecomposable left ideals. In due course, several indecomposable $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules will emerge, among these the modules $\mathbf{P}_\ell$, which have the property that if

$$0 \to L \to E \to \mathbf{P}_\ell \to 0 \qquad \text{and} \qquad 0 \to \mathbf{P}_\ell \to F \to M \to 0$$

are short exact sequences of $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules, then $\mathbf{P}_\ell$ embeds as a direct summand into E and into F. The algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ exemplifies many useful concepts from algebra: the Jacobson radical, Loewy layers, the Cartan matrix, and so on. Furthermore, $\mathfrak{U}_q(\mathfrak{sl}_2)$ nicely illustrates the multiplicity relations pertaining to Frobenius algebras.