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Fredholm Determinants, Differential Equations and Matrix Models

Craig A. Tracy[1], **Harold Widom**[2]

[1] Department of Mathematics and Institute of Theoretical Dynamics, University of California, Davis, CA 95616, U.S.A. E-mail address: tracy@itd.ucdavis.edu

[2] Department of Mathematics, University of California, Santa Cruz, CA 95064, USA. E-mail address: widom@cats.ucsc.edu

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Abstract: Orthogonal polynomial random matrix models of $N \times N$ hermitian matrices lead to Fredholm determinants of integral operators with kernel of the form $(\varphi(x)\psi(y)-\psi(x)\varphi(y))/x-y$. This paper is concerned with the Fredholm determinants of integral operators having kernel of this form and where the underlying set is the union of intervals $J=\bigcup_{j=1}^{m}(a_{2j-1}, a_{2j})$. The emphasis is on the determinants thought of as functions of the end-points a_k.

We show that these Fredholm determinants with kernels of the general form described above are expressible in terms of solutions of systems of PDE's as long as φ and ψ satisfy a certain type of differentiation formula. The (φ, ψ) pairs for the sine, Airy, and Bessel kernels satisfy such relations, as do the pairs which arise in the finite N Hermite, Laguerre and Jacobi ensembles and in matrix models of 2D quantum gravity. Therefore we shall be able to write down the systems of PDE's for these ensembles as special cases of the general system.

An analysis of these equations will lead to explicit representations in terms of Painlevé transcendents for the distribution functions of the largest and smallest eigenvalues in the finite N Hermite and Laguerre ensembles, and for the distribution functions of the largest and smallest singular values of rectangular matrices (of arbitrary dimensions) whose entries are independent identically distributed complex Gaussian variables.

There is also an exponential variant of the kernel in which the denominator is replaced by $e^{bx}-e^{by}$, where b is an arbitrary complex number. We shall find an analogous system of differential equations in this setting. If $b=i$ then we can interpret our operator as acting on (a subset of) the unit circle in the complex plane. As an application of this we shall write down a system of PDE's for Dyson's circular ensemble of $N \times N$ unitary matrices, and then an ODE if J is an arc of the circle.

I. Introduction

It is a fundamental result of Gaudin and Mehta that orthogonal polynomial random matrix models of $N \times N$ hermitian matrices lead to integral operators