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Nonintegrability of the Classical Zeeman Hamiltonian

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Abstract: We prove that the Hamiltonian H of the three dimensional hydrogen atom in a uniform static magnetic field $\mathbf{B}$ does not have an integral which (i) is real analytic on the phase space $\mathscr{M}$ of the system; (ii) is in involution with the component M_3 of the angular momentum along $\mathbf{B}$; (iii) is functionally independent of H and M_3 and (iv) has a meromorphic (single-valued) extension to the complexification of $\mathscr{M}$ in $\mathbb{C}^6$. This follows from the fact that the Hamiltonian K_M of two degrees of freedom obtained by fixing M_3 at certain nonzero values M and reducing H w.r. to the rotational symmetry about the magnetic field, has a complexification which is nonintegrable in the Ziglin sense. We prove this nonintegrability by demonstrating that for each such M the monodromy group of the normal variational equation along a certain complexified phase curve of K_M is not Ziglin, using Churchill and Rod's adaptation of Kovacic's algorithm to the Ziglin analysis. Analogous arguments prove that the Hamiltonian of the Størmer problem is nonintegrable in the same sense.

I. Introduction

There is intense interest in the classical and quantum mechanical behavior of the hydrogen atom in a static magnetic field (the "magnetized hydrogen atom" for short), and the physics literature on this subject is enormous (see e.g., [1–5] and the bibliographical information contained therein). The topic is important both from a fundamental physical viewpoint and because of its atomic physics and astrophysics applications. From a fundamental point of view, a large part of interest in the classical magnetized hydrogen atom arises because of the belief among many physicists (supported by numerous numerical and formal analytical arguments) that chaos in classical systems entails chaos, in some sense, in the corresponding

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