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Morse and Melnikov Functions for NLS Pde's

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Abstract. The theory of the focusing NLS equation under periodic boundary conditions, together with the Floquet spectral theory of its associated Zakharov-Shabat linear operator $\hat{L}$, is developed in sufficient detail for later use in studies of perturbations of the NLS equation. "Counting lemmas" for the non-selfadjoint operator $\hat{L}$, are established which control its spectrum and show that all of its eccentricities are finite in number and must reside within a finite disc D in the complex eigenvalue plane. The radius of the disc D is controlled by the H^1 norm of the potential $\vec{q}$. For this integrable NLS Hamiltonian system, unstable tori are identified, and Backlund transformations are then used to construct global representations of their stable and unstable manifolds – "whiskered tori" for the NLS pde.

The Floquet discriminant $\Delta(\lambda;\vec{q})$ is used to introduce a natural sequence of NLS constants of motion, $[\mathbf{F}_j(\vec{q}) \equiv \Delta(\lambda = \lambda_j^c(\vec{q});\vec{q})$, where λ_j^c denotes the j^{th} critical point of the Floquet discriminant $\Delta(\lambda)]$. A Taylor series expansion of the constants $\mathbf{F}_j(\vec{q})$, with explicit representations of the first and second variations, is then used to study neighborhoods of the whiskered tori. In particular, critical tori with hyperbolic structure are identified through the first and second variations of $\mathbf{F}_j(\vec{q})$, which themselves are expressed in terms of quadratic products of eigenfunctions of $\hat{L}$. The second variation permits identification, within the disc D, of important bifurcations in the spectral configurations of the operator $\hat{L}$. The constant $\mathbf{F}_j(\vec{q})$, as the height of the Floquet discriminant over the critical point λ_j^c, admits a natural interpretation as a Morse function for NLS isospectral level sets. This Morse interpretation is studied in some detail. It is valid globally for the infinite tail, $\{\mathbf{F}_j(\vec{q})\}_{|j|>N}$, which is associated with critical points outside the disc D. Within this disc, the interpretation is only valid locally, with the same obstruction to its global validity as to a global ordering of the spectrum. Nevertheless, this local Morse theory, together with the Backlund representations of the whiskered tori, produces extremely clear pictures of the stratification of NLS invariant sets near these whiskered tori – pictures which are

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