Commun. Math. Phys. 161, 631–643 (1994)

Communications in
Mathematical
Physics

Universal Estimate of the Gap for the Kac Operator in the Convex Case

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Received: 29 July 1993

Abstract: The aim of this paper is to prove that if V is a strictly convex potential with quadratic behavior at ∞, then the quotient μ_2/μ_1 between the largest eigenvalue and the second eigenvalue of the Kac operator defined on $L^2(\mathbb{R}^m)$ by $\exp -V(x)/2 \cdot \exp \Delta_x \cdot \exp -V(x)/2$, where Δ_x is the Laplacian on $\mathbb{R}^m$ satisfies the condition:

$$\mu_2/\mu_1 \leq \exp - \cosh^{-1}(\sigma + 1)/2\,,$$

where σ is such that $\operatorname{Hess} V(x) \geq \sigma > 0$.

1. Introduction

In some problems in statistical mechanics on a lattice $\mathbb{Z}^2$, a mechanism of reduction to a one dimensional lattice permits to reduce the general questions about correlations or thermodynamic limit to corresponding spectral properties for a compact operator K_V associated to a C^∞ potential V by the formula:

$$K_V = \exp -V/2 \cdot \exp \Delta \cdot \exp -V/2\,,$$

where Δ is the usual Laplacian on $\mathbb{R}^m$. It was proved in [22], that in the case of the Schrödinger operator, the assumption that V is strictly convex uniformly in $\mathbb{R}^m$, that is satisfying for some $\sigma > 0$,

$$\inf_x (\operatorname{Hess} V)(x) = \sigma > 0\,, \tag{1.1}$$

permits to get a minoration of the splitting between the second eigenvalue λ_2 and the first eigenvalue λ_1:

$$\lambda_2 - \lambda_1 \geq \sqrt{2\sigma}\,. \tag{1.2}$$

This condition appears to be optimal in the case of the harmonic oscillator in the sense that we get equality. So it is natural to ask for the same type question in the case of the Kac operator. Under condition (1.1) (and some conditions on the derivatives),