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Low Energy Asymptotics for Schrödinger Operators with Slowly Decreasing Potentials

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Abstract: Low energy behavior of Schrödinger operators with potentials which decay slowly at infinity is studied. It is shown that if the potential is positive then the zero energy is very regular and the resolvent is smooth near 0. This implies rapid local decay for the solutions of the Schrödinger equation. On the other hand, if the potential is negative then the resolvent has discontinuity at zero energy. Thus one cannot expect local decay faster than order t^{-1} as $t \to \infty$.

1. Introduction

In this paper we consider the Schrödinger operator

$$H = H_0 + V(x) = -\hbar^2 \Delta + V(x) \quad \text{on} \quad L^2(\mathbf{R}^d), \quad d \geqq 1 .$$

We will assume $V(x) \sim c|x|^{-\rho}$ as $|x| \to \infty$, and study the behavior of $(H - z)^{-1}$ near $z = 0$. If $\rho > 2$ then V is called *very short range* and the behavior of $(H - z)^{-1}$ near $z = 0$ was studied by Jensen, Kato and others (see, e.g., [JK, J, Mu]). If $d = 3$ and ρ is sufficiently large, then it is known that $(H - z)^{-1}$ has an asymptotic expansion in $z^{1/2}$:

$$(H - z)^{-1} \sim B_{-2} z^{-1} + B_{-1} z^{-1/2} + B_0 z^0 + \cdots, \quad z \to 0 .$$

The top term B_{-2} comes from the 0-energy eigenvalue, and B_{-1} comes from the 0-energy resonance. Since they are unstable under small perturbations, $(H - z)^{-1}$ is generically regular near $z = 0$.

On the other hand, if $0 < \rho < 2$, then V is called *slowly decreasing*, and it is known that $(H - z)^{-1}$ behaves quite differently near $z = 0$. For the one-dimensional case, this problem was studied by Yafaev [Y1] in detail using integral equation techniques. For the higher dimensional case, Yafaev also studied Schrödinger operators with positive slowly decreasing potentials ([Y2]). In particular, he proved that $(1 + |x|)^{-\alpha} H^{-m} (1 + |x|)^{-\beta}$ is bounded in $L^2(\mathbf{R}^d)$ if $\alpha + \beta > m\rho$. Using this estimate, the low energy asymptotics of $(H - z)^{-1}$ was