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Brownian Motion with Restoring Drift: The Petit and Micro-Canonical Ensembles

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Abstract: Let $f(Q)$ be odd and positive near $+\infty$. Then the non-linear wave equation $\partial^2 Q/\partial t^2 - \partial^2 Q/\partial x^2 + f(Q) = 0$, considered on the circle $0 \leq x < L$, can be written in Hamiltonian form $Q^\bullet = \partial H/\partial P$, $P^\bullet = -\partial H/\partial Q$ with

$$P = Q^\bullet \quad \text{and} \quad H = \frac{1}{2}\int_0^L (Q')^2 + \int_0^L F(Q) + \frac{1}{2}\int_0^L P^2\,;$$

the corresponding flow preserves the (suitably interpreted) "petit ensemble" $e^{-H}d^\infty Q d^\infty P$; and, for $L \uparrow \infty, Q$ settles down to the stationary diffusion with infinitesimal operator $\frac{1}{2}\partial^2/\partial Q^2 + m(Q)\partial/\partial Q, m$ being the logarithmic derivative of the ground state of $-d^2/dQ^2 \mid F(Q)$. This diffusion is the "Brownian motion with restoring drift"; see McKean-Vaninsky [1993(1)]. For reasons suggested by the paper of Lebowitz-Rose-Speer [1988] on NLS, it is interesting to study the "micro-canonical ensemble" obtained by restricting to the sphere $\int_0^L Q^2 = N$ and making $L \uparrow \infty$ with fixed $D = N/L$. Now, for $F(Q)/Q^2 \to \infty$, the same type of diffusion appears, but with drift arising from the modified potential $F(Q) + cQ^2, c$ being chosen so that the mean of Q^2 is the assigned number D. The proof employs Döblin's method of "loops" [1937] and steepest descent. The same is true for $F(Q) = m^2Q^2$, only now the proof is elementary. The outcome is also the same if $F(Q)/Q^2 \to 0$, provided D is smaller than the petit canonical mean of Q^2; for D larger than this mean, the matter is more subtle and the outcome is unknown.

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