Commun. Math. Phys. 160, 457–473 (1994)

Communications in
Mathematical
Physics

Fractal Wavelet Dimensions and Localization

Matthias Holschneider
CPT-CNRS (Luminy), Case 907, F-13288 Marseille Luminy, France. e-mail: hols@marcptnx1.univ-mrs. fr

Received: 29 December 1992/in revised form: 1 June 1993

Abstract: In this paper we want to give a new definition of fractal dimensions as small scale behavior of the q-energy of wavelet transforms. This is a generalization of previous multi-fractal approaches. With this particular definition we will show that the 2-dimension (=correlation dimension) of the spectral measure determines the long time behavior of the time evolution generated by a bounded self-adjoint operator acting in some Hilbert space $\mathscr{H}$. It will be proved that for ϕ, $\psi \in \mathscr{H}$ we have

$$\liminf_{T\to\infty} \frac{\log \int_0^T d\omega |\langle \psi | e^{-iA\omega}\phi\rangle|^2}{\log T} = -\kappa^+(2)$$

and that

$$\limsup_{T\to\infty} \frac{\log \int_0^T d\omega |\langle \psi | e^{-iA\omega}\phi\rangle|^2}{\log T} = -\kappa^-(2)\,,$$

where $\kappa^{\pm}(2)$ are the upper and lower correlation dimensions of the spectral measure associated with ψ and ϕ. A quantitative version of the RAGE theorem shall also be given.

1. Introduction

Let μ be a finite (signed) measure. A well known theorem of Wiener states that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T d\omega |\hat{\mu}(\omega)|^2 = \sum_{x\in\mathbb{R}} |\mu\{x\}|^2\,,$$

where the Fourier transform is given by

$$\hat{\mu}(\omega) = \int d\mu(t)\, e^{-i\omega t}\,.$$

Note that the sum is finite since μ is finite. Now let A be a self-adjoint operator acting in some Hilbert-space $\mathscr{H}$. For any state $\phi \in \mathscr{H}$ we shall be interested in the