Commun. Math. Phys. 160, 81–92 (1994)

Communications in
Mathematical
Physics

On the Rate of Quantum Ergodicity I: Upper Bounds

Steven Zelditch★

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA

Received: 3 May 1993

Abstract: One problem in quantum ergodicity is to estimate the rate of decay of the sums

$$S_k(\lambda; A) = \frac{1}{N(\lambda)} \sum_{\sqrt{\lambda_j} \leqq \lambda} |(A\varphi_j, \varphi_j) - \bar{\sigma}_A|^k$$

on a compact Riemannian manifold (M, g) with ergodic geodesic flow. Here, $\{\lambda_j, \varphi_j\}$ are the spectral data of the Δ of (M, g), A is a 0-th order ψDO, $\bar{\sigma}_A$ is the (Liouville) average of its principal symbol and $N(\lambda) = \#\{j: \sqrt{\lambda_j} \leqq \lambda\}$. That $S_k(\lambda; A) = o(1)$ is proved in [S, Z.1, CV.1]. Our purpose here is to show that $S_k(\lambda; A) = O((\log \lambda)^{-k/2})$ on a manifold of (possibly variable) negative curvature. The main new ingredient is the central limit theorem for geodesic flows on such spaces ([R, Si]).

Quantum ergodicity is the study of the spectral properties of Schrödinger operators with ergodic classical flows. In this paper, we will be concerned with a special case: that of a Laplacian on a compact n-dimensional Riemannian manifold M of negative curvature. As is well known, the geodesic flow G^t on S^*M is then ergodic. Δ is also quantum ergodic in the following sense: for any choice of orthonormal basis $\{\phi_j\}$ of eigenfunctions

$$\Delta\phi_j = \lambda_j\phi_j, \quad 0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \ldots \uparrow \infty$$

and any $A \in \Psi^0(M)$, one has

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\sqrt{\lambda_j} \leqq \lambda} |(A\varphi_j, \varphi_j) - \bar{\sigma}_A| = 0 \,. \tag{0.1}$$

Here, $N(\lambda) = \#\{j: \sqrt{\lambda_j} \leqq \lambda\}$, $\Psi^m(M)$ is the space of ψDO's (pseudodifferential operators) of order m, σ_A is the principal symbol and $\bar{\sigma}_A := \frac{1}{\text{vol}(S^*M)} \int_{S^*M} \sigma_A d\mu$ is

★ Partially supported by NSF grant #DMS-9243626