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Representations of Affine Lie Algebras, Elliptic r-Matrix Systems, and Special Functions

Pavel I. Etingof

Department of Mathematics, Yale University, New Haven, CT 06520, USA
e-mail etingof@pascal.math.yale.edu

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Abstract: The author considers an elliptic analogue of the Knizhnik–Zamolodchikov equations – the consistent system of linear differential equations arising from the elliptic solution of the classical Yang–Baxter equation for the Lie algebra $\mathfrak{sl}_N$. The solutions of this system are interpreted as traces of products of intertwining operators between certain representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_N$. A new differential equation for such traces characterizing their behavior under the variation of the modulus of the underlying elliptic curve is deduced. This equation is consistent with the original system.

It is shown that the system extended by the new equation is modular invariant, and the corresponding monodromy representations of the modular group are defined. Some elementary examples in which the system can be solved explicitly (in terms of elliptic and modular functions) are considered. The monodromy of the system is explicitly computed with the help of the trace interpretation of solutions. Projective representations of the braid group of the torus and representations of the double affine Hecke algebra are obtained.

Introduction

In 1984 Knizhnik and Zamolodchikov [KZ] studied matrix elements of products of intertwining operators between representations of the affinization $\hat{\mathfrak{g}}$ of a finite dimensional simple complex Lie algebra $\mathfrak{g}$ at level k. These matrix elements are analytic functions of several complex variables, and it was found that they satisfy a certain remarkable system of linear differential equations which is now called the Knizhnik–Zamolodchikov (KZ) system:

$$\kappa \frac{\partial \Psi}{\partial z_i} = \sum_{j=1, j \neq i}^{n} \frac{\Omega_{ij}}{z_i - z_j} \Psi . \tag{1}$$

Here $\Psi(z_1, \ldots, z_n)$ is a function of n complex variables with values in the product $W = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ of n representations of $\mathfrak{g}$, κ is a nonzero complex number,