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An Identification of the Connections of Quillen and Beilinson–Schechtman

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Abstract: Given a family of Riemann surfaces and a holomorphic vector bundle Beilinson and Schechtman construct a canonical connection on the associated determinant bundle. We prove the conjecture which states that their connection coincides with the Quillen connection. This is done by reducing to the case where $\bar{\partial}$ along fibers are invertible. Both connection forms become more accessible in this case.

Introduction

Let π: $X \to S$ be the parametrization of a family of compact Riemann surfaces, $E \to X$ a holomorphic vector bundle, and $\lambda_E = \det(R\pi_* E)$ the determinant bundle over S. Given C^∞ connections on $T_{X/S}$ and E, not necessarily arising from metrics, Beilinson and Schechtman [BS, §5] construct a formal parametrix $p(z, \zeta)$ for $\bar{\partial}_z$ ($\bar{\partial}$ along the fibers) and using p they derive a (1, 0) connection ∇_{BS} on λ_E. These are given by local formulas in terms of $\nabla_{T_{X/S}}$ and ∇_E and fiber integrals. Local calculations show that the (1, 1) curvature $\bar{\partial}_S \nabla_{\mathrm{BS}}$ is equal as a differential form to the fiber integral prescribed by the Grothendieck Riemann Roch formula. The only nonelementary part of the connection ∇_{BS} is in identifying it with splittings of complexes on X, which depends on the relative duality theorem.

On λ_E^{-1} of course there is the well known Quillen metric and the associated Quillen connection ∇_{Q} determined by metrics on $T_{X/S}$ and E. For locally Kähler families of compact complex manifolds the form level Grothendieck Riemann Roch for curvature of ∇_{Q} is known [BF, BGS]. Beilinson–Schechtman remark in [BS, 5.6]: "It seems very probable" that when ∇_E, $\nabla_{T_{X/S}}$ arise from hermitian metrics on E, $T_{X/S}$, then ∇_{BS} is just the connection associated to the Quillen metric for λ_E.

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