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Solutions with High Dimensional Singular Set, to a Conformally Invariant Elliptic Equation in $\mathbb{R}^4$ and in $\mathbb{R}^6$

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Abstract: We construct positive weak solutions to the equation $-\Delta_0 v = v^{\frac{n+2}{n-2}}$, where $-\Delta_0$ denotes the conformal Laplacian on the n-sphere ($n = 4, 6$), having singular sets of Hausdorff dimension greater than or equal to $\frac{n-2}{2}$.

1. Introduction

In their paper, Schoen and Yau have stated the following conjecture:

Conjecture [8]: All positive weak solutions of $-\Delta_0 v = v^{\frac{n+2}{n-2}}$, with $v \in L^{\frac{n+2}{n-2}}(\mathbb{S}^n)$, have singular set of Hausdorff dimension less than or equal to $(n-2)/2$. Here $-\Delta_0$ denotes the conformal Laplacian for the standard metric on the sphere $\mathbb{S}^n$.

This problem can be formulated in $\mathbb{R}^n$ as follows [9]: We define the measure $d\mu = (1 + |x|^2)^{-n} dx$ on $\mathbb{R}^n$. Assume that $u \in L^{\frac{n+2}{n-2}}(\mathbb{R}^n, d\mu)$ is a weak positive solution of

$$-\Delta u = u^{\frac{n+2}{n-2}} . \tag{1}$$

Then, the Hausdorff dimension of the singular set if u is less than or equal to $(n-2)/2$.

Many attempts have been made to find solutions of (1) with a prescribed singular set. In a very difficult paper [7], Schoen builds solutions of (1) with prescribed isolated singularities. In another paper [8], Schoen and Yau have used the geometrical meaning of Eq. (1) in order to derive, through ideas of conformal geometry, the existence of singular solutions having a singular set whose Hausdorff dimension is less than or equal to $(n-2)/2$. More recently Mazzeo and Smale have proved in [4] the existence of solutions of (1) singular over some manifold which is a small deformation of a sphere $\mathbb{S}^k$, with $k < (n-2)/2$. Their method is based on the study of degenerate operators.

In this paper, we give some counter-examples to the conjecture stated above when $n = 4$ and when $n = 6$. More precisely, we prove the result: