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Vertex Representations for *N*-Toroidal Lie Algebras and a Generalization of the Virasoro Algebra

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Abstract: Vertex representations are obtained for toroidal Lie algebras for any number of variables. These representations afford representations of certain n-variable generalizations of the Virasoro algebra that are abelian extensions of the Lie algebra of vector fields on a torus.

0. Introduction

In this paper we construct faithful vertex operator representations for the universal central extension τ_n of $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $\mathfrak{g}$ is a simple, simply-laced finite dimensional Lie algebra over $\mathbb{C}$. We call τ_n the n-toroidal Lie algebra. These representations also afford representations for an abelian extension of the Lie algebra of derivations of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. This latter Lie algebra is a generalization of the Virasoro algebra, and so this whole construction is a generalization of both the Frenkel–Kac and the Segal–Sugawara constructions which are well known for the case $n = 1$.

For a suitable non-degenerate integral lattice Γ and an even integral sublattice Q (cf. Sect. 3), we construct the Fock space $V(\Gamma, \mathfrak{b}) = \mathbb{C}[\Gamma] \otimes S(\mathfrak{b}_-)$, where $\mathfrak{b}$ is a Heisenberg algebra defined by Γ. For each α in Q we define vertex operator $X(\alpha, z)$ (cf. 3.7) such that its Fourier components $X_n(\alpha)$ act on $V(\Gamma, \mathfrak{b})$. Our first result (Theorem 3.14) says that the Lie algebra generated by operators $X_k(\alpha)$ ($\alpha \in Q, (\alpha \mid \alpha) = 2$) is isomorphic to $\tau_{[n]}$. We also prove that the "zero moments" (taking $k = 0$ above) generate the Lie algebra $\tau_{[n-1]}$ (Theorem 3.17).

Theorem 3.14 in the case $n = 1$ is due to Frenkel–Kac [FK] and $\tau_{[1]}$ is the non-twisted affine Lie algebra. The case $n = 2$ is due to [MEY]. Our method of proof here differs considerably from that of [MEY]. It is more explicit in the sense that we give operators for every vector of $\tau_{[n]}$ and prove that the necessary commutators hold. For example the vector $h \otimes t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$ in $\tau_{[n]}$ is represented by the operator $T_{r_n}^h(\delta_{\underline{r}})$ (cf. 3.10) which is not clear from [MEY]. A significant difference is the

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