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Integrable Operator Representations of $\mathbb{R}_q^2$, $X_{q,\gamma}$ and $SL_q(2, \mathbb{R})$

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Abstract: Let q be a complex number such that $|q| = 1$ and $q^4 \neq 1$. Integrable ("well-behaved") operator representations of the $*$-algebra $SL_q(2, \mathbb{R})$ in Hilbert space are defined and completely classified up to unitary equivalence. In order to do this, the relation $xy - qyx = \gamma(1 - q), \gamma \in \mathbb{R}$, for self-adjoint operators x and y is studied in detail. Integrable representations for this relation are defined and classified.

0. Introduction

The study of non-compact quantum groups or more generally of non-compact quantum spaces at the Hilbert space level leads to new features and difficulties which do not occur in the compact case. The main source of these problems is the fact that the generators of the "ring of functions on the non-compact quantum space" do not have (enough) representations by bounded operators in general. On the technical level, we are concerned with (finitely many) *unbounded* operators which satisfy the commutation relations from the definition of the quantum space. The first problem that arises is to select the "well-behaved" representations for this set of relations. Following the terminology commonly used in representation theory of Lie algebras and of general $*$-algebras (see e.g. [J, S1]), we call these representations "integrable." In general, there is no canonical way to define integrability for a given set of commutation relations. The main purpose of this paper is to define and to classify integrable representations for the real quantum vector space $\mathbb{R}_q^2$ (i.e. for the relation $xy = qyx$), for the real quantum hyperboloid $X_{q,\gamma}$ (i.e. for the relation $xy - qyx = \gamma(1 - q), \gamma \in \mathbb{R}/\{0\}$) and for the real form $SL_q(2, \mathbb{R})$ of the quantum group $SL_q(2)$, where $x = x^*$, $y = y^*$ and $|q| = 1$, $q^4 \neq 1$. Our main aim was to investigate $SL_q(2, \mathbb{R})$, but it turned out immediately that this requires a very detailed treatment of both $\mathbb{R}_q^2$ and $X_{q,\gamma}$. Knowing the irreducible integrable representations could be a starting point for studying some "analysis" on the quantum group $SL_q(2, \mathbb{R})$. The problem of defining integrability for certain operator relations was touched in [D] and in [W1]. It was studied in [OS1, OS2 and S2]. If the relations are "nice," it may happen that for *irreducible* integrable representations all