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Level-Spacing Distributions and the Airy Kernel

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Abstract: Scaling level-spacing distribution functions in the "bulk of the spectrum" in random matrix models of $N \times N$ hermitian matrices and then going to the limit $N \to \infty$ leads to the Fredholm determinant of the *sine kernel* $\sin \pi(x-y)/\pi(x-y)$. Similarly a scaling limit at the "edge of the spectrum" leads to the *Airy kernel* $[\mathrm{Ai}(x)\,\mathrm{Ai}(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)]/(x-y)$. In this paper we derive analogues for this Airy kernel of the following properties of the sine kernel: the completely integrable system of P.D.E.'s found by Jimbo, Miwa, Môri, and Sato; the expression, in the case of a single interval, of the Fredholm determinant in terms of a Painlevé transcendent; the existence of a commuting differential operator; and the fact that this operator can be used in the derivation of asymptotics, for general n, of the probability that an interval contains precisely n eigenvalues.

I. Introduction and Summary of Results

A. Introduction

In this paper we present new results for the *level spacing distribution functions* obtained from scaling random matrix models of $N \times N$ hermitian matrices at the edge of the support of the (tree-level) eigenvalue densities when the parameters of the potential V are not "finely tuned." This universality class is already present in the Gaussian Unitary Ensemble. It is known [3, 10, 20, 23] that these distribution functions are expressible in terms of a Fredholm determinant of an integral operator K whose kernel involves Airy functions.

It turns out that there are striking analogies between the properties of this *Airy kernel*,

$$K(x,y) = \frac{A(x)A'(y) - A'(x)A(y)}{x-y},$$

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