Commun. Math. Phys. 159, 15–27 (1994)

Communications in
**Mathematical
Physics**

Super-derivations

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Received November 25, 1992

Abstract: It is shown that the square of a super-derivation can never be a generator (without taking its closure) if it is unbounded and self-adjoint.

1. Introduction

The notion of a quantum algebra has been introduced by A. Jaffe et al. [5–7] in connection with entire cyclic cohomology (cf. [3, 4, 8]). A key ingredient to this notion is a super-derivation, defined on a graded C^*-algebra, whose square *is* or *extends to* the generator of a one-parameter group of *-automorphisms. In this note we study the relationship between the super-derivation and the generator to seek the *right* definition of a quantum algebra and obtain among others the result stated in the abstract, i.e., if the square of a self-adjoint super-derivation is a generator then it is bounded.

We will state the main results in Sect. 2 and give their proofs in Sects. 3–6. Finally we will give a *spatial* example based on the algebra of bounded operators on a Hilbert space. One of the authors (A.K.) is grateful to C.J. K. Batty for many discussions.

In the rest of this section we will state the definition of a super-derivation and give some basic properties.

Let (A,γ) be a graded C^*-algebra; i.e., A is a C^*-algebra and γ is a *-automorphism of A of period two. Let

$$A_e = \{a \in A \mid \gamma(a) = a\},\ A_o = \{a \in A \mid \gamma(a) = -a\}\ .$$

Then it follows that A_e is a sub-C^*-algebra of A and that $A_e A_o \supset A_o$, $A_o^* = A_o$, and $A_o A_o \subset A_e$. The C^*-algebra A is the direct sum of A_e and A_o as a Banach space.

Let d be a super-derivation of A; i.e., its domain $D(d)$ is a (dense) γ-invariant subalgebra of A and d is a linear map of $D(d)$ into A such that

$$d(ab) = da \cdot b + \gamma(a) \cdot db,\ a, b \in D(d)\ ,$$

and $\gamma \circ d = -d \circ \gamma$. In particular

$$D(d) = D(d) \cap A_e + D(d) \cap A_o$$