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Strict Inequality for Critical Values of Potts Models and Random-Cluster Processes

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Abstract: We prove that the critical value β_c of a ferromagnetic Potts model is a *strictly* decreasing function of the strengths of interaction of the process. This is achieved in the (more) general context of the random-cluster representation of Fortuin and Kasteleyn, by deriving and utilizing a formula which generalizes the technique known in percolation theory as Russo's formula. As a byproduct of the method, we present a general argument for showing that, at any given point on the critical surface of a multiparameter process, the values of a certain critical exponent do not depend on the direction of approach of that point. Our results apply to all random-cluster processes satisfying the FKG inequality.

1. Introduction

In the general study of phase transitions, it is commonly the case that the value of the critical point is a *monotone* function of the strengths of local interactions (and hence of the structure of the underlying graph). For example, the critical probability of a percolation process cannot increase if new edges are added to the lattice. Similarly, the critical temperature of an Ising model cannot decrease if the intensity of any pair-interaction is increased. Such monotonicity, when true, is usually easy to prove. *Strict* monotonicity, on the other hand, presents new difficulties. A general approach to the problem of proving *strict* monotonicity of the critical point was described by Aizenman and Grimmett (1991). Using this approach, the latter authors obtained necessary and sufficient conditions for strict monotonicity in percolation, and in addition proved strict monotonicity for Ising models with pair-interactions. In this paper, we prove such a result for Potts models, thereby generalizing the latter conclusion of Aizenman and Grimmett (1991).

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