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Quasifinite Highest Weight Modules over the Lie Algebra of Differential Operators on the Circle

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Abstract: We classify positive energy representations with finite degeneracies of the Lie algebra $W_{1+\infty}$ and construct them in terms of representation theory of the Lie algebra $\widehat{gl}(\infty, R_m)$ of infinites matrices with finite number of non-zero diagonals over the algebra $R_m = \mathbb{C}[t]/(t^{m+1})$. The unitary ones are classified as well. Similar results are obtained for the sin-algebras.

0. Introduction

0.1. Recent progress in conformal field theory revealed some unusual mathematical objects called the W_n-algebras [Z]. These algebras turned out to be quantizations of the second Gelfand-Dickey structure for Lax equations [FL]. The complicated structure of these algebras is greatly simplified in the limit $n = \infty$, the limiting algebra being the Lie algebra $\widehat{\mathscr{D}}$, the universal central extension of the Lie algebra of differential operators on the circle [KP]. (Physicists denote this Lie algebra by $W_{1+\infty}$ [PSR].) The possibility to get W_n from $\widehat{\mathscr{D}}$ has been studied in [R, RV]. A complete picture for classical W_n was obtained in [KhZ].

The main goal of the present paper is to classify and describe the irreducible quasifinite highest weight representation of the Lie algebra $\widehat{\mathscr{D}}$. The basic technical tool is the analytic completion $\widehat{\mathscr{D}}^{\mathscr{C}}$ of $\widehat{\mathscr{D}}$ and a family of its homomorphisms onto the central extension of the Lie algebra $\widetilde{gl}(\infty, R_m)$ of infinite matrices with finitely many non-zero diagonals over the ring $R_m = \mathbb{C}[t]/(t^{m+1})$.

The Lie algebra $\widehat{\mathscr{D}}$ may be obtained via a general construction (explained in Sect. 1) as a twisted Laurent polynomial algebra over the polynomial algebra $\mathbb{C}[w]$. It is easy to see that the only other Lie algebras obtained by this construction from $\mathbb{C}[w]$ are Lie algebras $\widehat{\mathscr{D}_q}$, the central extension of the Lie algebra of difference

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