Commun. Math. Phys. 157, 389–404 (1993)

Communications in
Mathematical
Physics

Moment Maps at the Quantum Level

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Received December 8, 1992; in revised form January 22, 1993

Abstract. We introduce the notion of moment maps for quantum groups acting on their module algebras. When the module algebras are quantizations of Poisson manifolds, we prove that the construction at the quantum level is a quantization of that at the semi-classical level. We also prove that the corresponding smashed product algebras are quantizations of the semi-direct product Poisson structures.

1. Introduction

The concept of moment maps for Hamiltonian actions is a very important one in symplectic geometry. In particular, it is a crucial tool in the study of symplectic reduced spaces, on which the reduced Hamiltonian systems live [Ms-We]. In the theory of Hopf algebras, a very important notion is that of inner actions (see [B-C-M] and the references therein). In the first part of this paper, we relate these two concepts in the two different fields. We show that by taking semi-classical limits, inner actions of Hopf algebras give rise to Poisson actions of Poisson groups with moment maps. This leads to the definition of moment maps for quantum group actions, the first step in carrying out quantum reduction.

An equally important concept in Hopf algebra theory is that of crossed products [B-C-M]. We show that in symplectic and Poisson geometry, this corresponds to semi-direct products of Poisson manifolds and Poisson groups.

Recall that if P is a symplectic manifold equipped with an action of a group G preserving the symplectic structure, then the space of G-invariant functions on P is closed under the Poisson bracket on functions on P, and thus the quotient space P/G, when it is a manifold, has a naturally defined Poisson structure. Flows of G-invariant Hamiltonians can be considered as living on P/G. More precisely, they live on the symplectic leaves in P/G. When the action is generated by a moment map $\phi: P \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G, the reduction procedure

★ Research partially supported by NSF grant DMS-89-07710