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Localization at Large Disorder and at Extreme Energies: An Elementary Derivation*

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Abstract. The work presents a short proof of localization under the conditions of either strong disorder ($\lambda > \lambda_0$) or extreme energies for a wide class of self adjoint operators with random matrix elements, acting in ℓ^2 spaces. A prototypical example is the discrete Schrödinger operator $H = -\Delta + U_0(x) + \lambda V_x$ on Z^d, $d \geqq 1$, with $U_0(x)$ a specified background potential and $\{V_x\}$ generated as random variables. The general results apply to operators with $-\Delta$ replaced by a non-local self adjoint operator T whose matrix elements satisfy: $\sum_y |T_{x,y}|^s \leqq$ Const., uniformly in x, for some $s < 1$. Localization means here that within a specified energy range the spectrum of H is of the pure-point type, or equivalently – the wave functions do not spread indefinitely under the unitary time evolution generated by H. The effect is produced by strong disorder in either the potential or in the off-diagonal matrix elements $T_{x,y}$. Under rapid decay of $T_{x,y}$, the corresponding eigenfunctions are also proven to decay exponentially. The method is based on resolvent techniques. The central technical ideas include the use of low moments of the resolvent kernel, i.e., $\langle |G_E(x,y)|^s \rangle$ with s small enough (<1) to avoid the divergence caused by the distribution's Cauchy tails, and an effective use of the simple form of the dependence of $G_E(x,y)$ on the individual matrix elements of H in elucidating the implications of the fundamental equation $(H-E)G_E(x,x_0) = \delta_{x,x_0}$. This approach simplifies previous derivations of localization results, avoiding the small denominator difficulties which have been hitherto encountered in the subject. It also yields some new results which include localization under the following sets of conditions: i) potentials with an inhomogeneous non-random part $U_0(x)$, ii) the Bethe lattice, iii) operators with very slow decay in the off-diagonal terms ($T_{x,y} \approx 1/|x-y|^{(d+\varepsilon)}$), and iv) localization produced by disordered boundary conditions.

Contents

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