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Singular Measures in Circle Dynamics

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Abstract. Critical circle homeomorphisms have an invariant measure totally singular with respect to the Lebesgue measure. We prove that singularities of the invariant measure are of Hölder type. The Hausdorff dimension of the invariant measure is less than 1 but greater than 0.

1. Preliminaries

1.1. Discussion of the Results. The long time behavior of nonlinear dynamical systems can be often characterized by means of invariant measures. A variety of "multifractal formalisms" have been developed recently to study statistical properties of singular measures (see [4, 2] for more details) which appear as a natural description of many physical phenomena. One of the characteristic quantities describing the multifractal structure of a singular measure μ is a singularity spectrum $g(\alpha)$ which is usually defined in an informal way (see [4, 2] and many others) as follows:

Cover the support of μ by small boxes L_i of size l. Then define the singularity strength α_i of μ in the i^{th} box by the relation:

$$\mu(L_i) \sim l^{\alpha_i} .$$

We count the number of boxes $N(\alpha)$ where μ has singularity strength between α and $\alpha + d\alpha$ (whatever that is to mean). Then $g(\alpha)$ is defined by the requirement that

$$N(\alpha) \sim l^{g(\alpha)} .$$

Unfortunately, many "multifractal formalisms" suffer from mathematical ambiguities (see [2] for a fuller discussion of this problem; for example, is $g(\alpha)$ a Hausdorff or a box dimension or something else?) even if they provide qualitative information on a given dynamical system. In the present paper we would like to propose a method of describing the dynamics of critical circle homeomorphisms.

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