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Ising Model and $N = 2$ Supersymmetric Theories

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Abstract. We establish a direct link between massive Ising model and arbitrary massive $N = 2$ supersymmetric QFT's in two dimensions. This explains why the equations which appear in the computation of spin-correlations in the non-critical Ising model are the same as those describing the geometry of vacua in $N = 2$ theories. The tau-function appearing in the Ising model (i.e., the spin correlation function) is reinterpreted in the $N = 2$ context as a new "index". In special cases this new index is related to the Ray–Singer analytic torsion, and can be viewed as a generalization of that to the loop space of Kähler manifolds.

1. Introduction

The two dimensional Ising model was among the first integrable models to be studied [1]. It is a very rich theory and has played a prominent role in the development of integrable hierarchies. In particular the notion of a tau function arose as a result of studying this theory. This line of development arose by the realization [1] that for the non-critical Ising model, which is equivalent to free massive fermions, spin correlations can be computed by relating them to solutions of certain differential equations. For example, for the two point function the corresponding equation turns out to be a special case of Painlevé III. Formalizing these ideas and generalizing them led the Japanese school [2] to a set of differential equations in n variables, whose solutions could be used to construct the n point spin correlation function for the massive free fermion theory; the first example of a *tau* function. From this point they went on to develop the more general notion of a tau function.

More recently, in a seemingly unrelated development, in studying the geometry of vacua of two dimensional $N = 2$ supersymmetric quantum field theories we encountered a set of equations [3] which captures Berry's curvature for the vacuum subsector of the theory. These equations which were derived using the $N = 2$ algebra and relied heavily on the topological nature of these theories were called *topological-anti-topological* or tt* equations. Some aspects of these computations were recently reinterpreted [4] as a way to define and compute a new