Commun. Math. Phys. 156, 527–546 (1993)

Communications in Mathematical Physics

Invariant Connections and Vortices

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Received February 21, 1992; in revised form August 31, 1992

Abstract. We study the vortex equations on a line bundle over a compact Kähler manifold. These are a generalization of the classical vortex equations over $\mathbb{R}^2$. We first prove an invariant version of the theorem of Donaldson, Uhlenbeck and Yau relating the existence of a Hermitian–Yang–Mills metric on a holomorphic bundle to the stability of such a bundle. We then show that the vortex equations are a dimensional reduction of the Hermitian–Yang–Mills equation. Using this fact and the theorem above we give a new existence proof for the vortex equations and describe the moduli space of solutions.

Introduction

In this paper we shall study a direct generalization of the vortex equations on $\mathbb{R}^2$ in which the euclidean plane is replaced by a compact Kähler manifold.

The vortex equations on $\mathbb{R}^2$ were first introduced in 1950 by Ginsburg and Landau [9] in the study of superconductivity. Geometrically they correspond to the equations satisfied by the absolute minima of the Yang–Mills–Higgs functional, defined for a unitary connection A and a smooth section ϕ of a Hermitian line bundle over $\mathbb{R}^2$ as

$$\mathrm{YMH}(A,\phi)=\int_{\mathbb{R}^2}|F_A|^2+|d_A\phi|^2+\frac{1}{4}(1-|\phi|^2)^2\,.$$

Here F_A is the curvature of A and $d_A\phi$ is the covariant derivative of ϕ.

If we regard $\mathbb{R}^2$ as the complex plane we may decompose with respect to the complex structure to get $d_A=d'_A+d''_A$. Then by integration by parts we can show that the functional above is bounded below by $2\pi d$, where d is an integer called the *vortex number*, and this minimum is attained if and only if

$$\left.\begin{aligned} d''_A\phi&=0\\ F_A&=\frac{1}{2}*(1-|\phi|^2)\end{aligned}\right\}\,.$$

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