Commun. Math. Phys. 156, 435–472 (1993)

Communications in Mathematical Physics

Chern–Simons Theory with Finite Gauge Group

Daniel S. Freed[1] **and Frank Quinn**[2]

[1] Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA
[2] Department of Mathematics, Virginia Polytechnical Institute, Blacksburg, VA 24061, USA

Received June 3, 1991; in revised form December 15, 1992

Abstract. We construct in detail a 2 + 1 dimensional gauge field theory with finite gauge group. In this case the path integral reduces to a finite sum, so there are no analytic problems with the quantization. The theory was originally introduced by Dijkgraaf and Witten without details. The point of working it out carefully is to focus on the algebraic structure, and particularly the construction of quantum Hilbert spaces on closed surfaces by cutting and pasting. This includes the "Verlinde formula." The careful development may serve as a model for dealing with similar issues in more complicated cases.

A typical course in quantum field theory begins with a thorough examination of a "toy model," usually the ϕ^4 theory. Our purpose here is to provide a detailed description of a "toy model" for *topological* quantum field theory, suitable for use as a foundation for more sophisticated developments. We carry through all the steps of the path integral quantization: start with a lagrangian, construct the classical action, construct a measure, and do the integral. When the gauge group is finite the "path integral" reduces to a finite sum. This remark makes it clear that the analytical difficulties simplify enormously, and that there should be no essential problem in carrying out the process. Many interesting features remain, however. The algebraic and topological structure are essentially unchanged, and are much clearer when not overshadowed by the analysis. And even the analysis does not entirely disappear: the details of the construction of the state spaces requires a much more precise formulation of the classical theory than is usually given, and reveals some incompleteness in the understanding of the classical theory for continuous Lie groups [F1].

Chern–Simons theory with finite gauge group was introduced by Dijkgraaf and Witten [DW], who essentially cataloged the possible lagrangians and gave some

The first author is supported by NSF grant DMS-8805684, an Alfred P. Sloan Research Fellowship, a Presidential Young Investigators award, and by the O'Donnell Foundation. The second author is supported by NSF grant DMS-9207973