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The Efimov Effect. Discrete Spectrum Asymptotics

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Abstract. We study a three-particle Schrödinger operator H for which none of the two-particle subsystems has negative bound states and at least two of them have zero energy resonances. We prove that under this condition the number $N(z)$ of bound states of H below $z < 0$ has the asymptotics $N(z) \sim \mathfrak{A}_0 |\log|z||$ as $z \to -0$, where the coefficient $\mathfrak{A}_0$ depends only on the ratio of masses of the particles.

1. Introduction

We are going to discuss the following remarkable phenomenon of the spectral theory of the three-body Schrödinger operators, known as the Efimov effect. Let h_α, $\alpha = 1, 2, 3$, be Hamiltonians describing two-particle subsystems of a three-particle system with the internal short-range potentials $v_\alpha(x)$, $x \in \mathbb{R}^3$. Suppose that none of h_α has negative eigenvalues and at least two of the hamiltonians h_α have zero energy resonances. Then the three-particle operator H will have infinitely many negative eigenvalues accumulating at zero. Below we denote by $N(z)$, $z < 0$, the number of eigenvalues of H lying on the left from the point z. For the first time the Efimov effect has been discussed in [4]. An independent proof on a physical level of rigor has been also given in [2]. The first rigorous proof has been presented in paper [12]. An alternative approach for spherically symmetric potentials v_α has been put forward in [10]. The growth of $N(z)$ as $z \to -0$ has been studied in paper [1] for the symmetric case. Namely, the authors of [1] have found the exponential asymptotics of eigenvalues corresponding to spherically symmetric bound states. This result is consistent with the lower bound

$$\liminf_{z \to -0} |\log|z||^{-1} N(z) > 0 , \tag{1.1}$$

established in [11] without any symmetry assumptions.

The aim of the present paper is to study the asymptotics of $N(z)$ as $z \to -0$. We do not assume that the pair potentials v_α are symmetric but suppose that $v_\alpha \leqq 0$.