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Global Solutions of the Relativistic Euler Equations

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Abstract. We demonstrate the existence of solutions with shocks for the equations describing a perfect fluid in special relativity, namely, $\operatorname{div} T = 0$, where $T^{ij} = (p + \rho c^2)u^i u^j + p\eta^{ij}$ is the stress energy tensor for the fluid. Here, p denotes the pressure, u the 4-velocity, ρ the mass-energy density of the fluid, η^{ij} the flat Minkowski metric, and c the speed of light. We assume that the equation of state is given by $p = \sigma^2 \rho$, where σ^2, the sound speed, is constant. For these equations, we construct bounded weak solutions of the initial value problem in two dimensional Minkowski spacetime, for any initial data of finite total variation. The analysis is based on showing that the total variation of the variable $\ln(\rho)$ is non-increasing on approximate weak solutions generated by Glimm's method, and so this quantity, unique to equations of this type, plays a role similar to an energy function. We also show that the weak solutions $(\rho(x^0, x^1), v(x^0, x^1))$ themselves satisfy the Lorentz invariant estimates $\operatorname{Var}\{\ln(\rho(x^0,\cdot)\} < V_0$ and $\operatorname{Var}\left\{\ln\dfrac{c + v(x^0,\cdot)}{c - v(x^0,\cdot)}\right\} < V_1$ for all $x^0 \geqq 0$, where V_0 and V_1 are Lorentz invariant constants that depend only on the total variation of the initial data, and v is the classical velocity. The equation of state $p = (c^2/3)\rho$ describes a gas of highly relativistic particles in several important general relativistic models which describe the evolution of stars.

1. Introduction

We consider the relativistic equations for a perfect fluid in Minkowski spacetime,

$$\operatorname{div} T = 0\,, \tag{1}$$

where

$$T^{ij} = (p + \rho c^2)u^i u^j + p\eta^{ij} \tag{2}$$

denotes the stress-energy tensor for the fluid. Recall that in Minkowski spacetime,

$$\operatorname{div} T \equiv T^i_{j,i}\,, \tag{3}$$