Commun. Math. Phys. 156, 17–36 (1993)

Communications in
Mathematical
Physics

Stable Equivalence of the Weak Closures of Free Groups Convolution Algebras

Florin Rădulescu

Department of Mathematics, U.C.L.A., Los Angeles, CA 90024, USA and I.H.E.S., 35 Route de Chartres, 91440 Bures-sur-Yvette, France. Present address: Department of Mathematics, University of California, Berkeley, CA 94720, USA. Permanent address: Institute of Mathematics of the Romanian Academy, Str. Academiei 14, Bucharest, Romania

Received January 20, 1992; in revised form October 22, 1992

Abstract. We prove in this paper that the von Neumann algebras associated to the free non-commutative groups are stably isomorphic, i.e. that they are isomorphic when tensorized by the algebra of all linear bounded operators on a separable, infinite dimensional Hilbert space. This gives positive evidence for an old question, due to R.V. Kadison (see also S. Sakai's book on W*-algebras), whether the von Neumann algebras associated to free groups are isomorphic or not.

In this paper we show that the algebras $\mathscr{L}(F_N) = \overline{\mathbb{C}(F_N)}^w$, the weak closures of the group algebras associated to free (nonabelian) groups F_N, $N \geqq 2$, N finite, are all stably isomorphic, i.e. that the isomorphism class of $\mathscr{L}(F_N) \otimes B(H)$ doesn't depend on $N \in \mathbb{N}$, $N \geqq 2$. (Here $B(H)$ is the algebra of bounded operators on an infinite dimensional, separable, Hilbert space H).

This may serve as evidence for an old problem of R.V. Kadison in the early '60's, on the isomorphism of the algebras $\mathscr{L}(F_N)$, $N \geqq 2$ ([2], see also [9] problem 4.4.44 and [3]).

The first remarkable breakthrough to this end, was the theorem of D. Voiculescu stating that

$$M_k(\mathbb{C}) \otimes \mathscr{L}(F_{(N-1)k^2+1})$$

is isomorphic to $\mathscr{L}(F_N)$, $k, N \in \mathbb{N}$, $N \geqq 2$. (This implies in particular that $\mathscr{L}(F_N)$ and $\mathscr{L}(F_{(N-1)k^2+1})$ are stably isomorphic for each $k, N \in \mathbb{N}$, $N \geqq 2$, but it doesn't imply, for example that $\mathscr{L}(F_2)$ and $\mathscr{L}(F_3)$ are stably isomorphic).

Our main tools will be the matrix representation for free families obtained by D. Voiculescu in the setting of noncommutative probability theory ([6, 7]), and the iterative technique of finding generators for reduced free algebras that we used in [4].

In terms of isomorphism classes of reduced algebras, the result of D. Voiculescu was stated as

$$\mathscr{L}(F_N)_{\frac{1}{k}} \cong \mathscr{L}(F_{(N-1)k^2+1}); \quad k, N \in \mathbb{N}, N \geqq 2 .$$