Commun. Math. Phys. 155, 385–413 (1993)

Communications in
Mathematical
Physics

Darboux Coordinates and Liouville-Arnold Integration in Loop Algebras*

M. R. Adams[1], **J. Harnad**[2,3] **and J. Hurtubise**[3,4]

[1] Department of Mathematics, University of Georgia, Athens, Georgia
[2] Department of Mathematics and Statistics, Concordia University, Montréal,Qué., Canada
[3] Centre de recherches mathématiques, Université de Montréal, C.P. 6128-A, Montréal, Qué., Canada
[4] Department of Mathematics, McGill University, Montréal, Qué., Canada

Received February 21, 1992; in revised form March 26, 1993

Abstract. Darboux coordinates are constructed on rational coadjoint orbits of the positive frequency part $\tilde{\mathfrak{g}}^+$ of loop algebras. These are given by the values of the spectral parameters at the divisors corresponding to eigenvector line bundles over the associated spectral curves, defined within a given matrix representation. A Liouville generating function is obtained in completely separated form and shown, through the Liouvile-Arnold integration method, to lead to the Abel map linearization of all Hamiltonian flows induced by the spectral invariants. As illustrative examples, the case $\mathfrak{g} = \mathfrak{sl}(2)$, together with its real forms, is shown to reproduce the classical integration methods for finite dimensional systems defined on quadrics, with the Liouville generating function expressed in hyperellipsoidal coordinates. For $\mathfrak{g} = \mathfrak{sl}(3)$, the method is applied to the computation of quasi-periodic solutions of the two component coupled nonlinear Schrödinger equation, a case which requires further symplectic constraints in order to deal with singularities in the spectral data at ∞.

Introduction

In [AHP, AHH1, AHH2], a unified approach was developed to the representation of both finite dimensional integrable Hamiltonian systems and quasi-periodic solutions of integrable PDE's as isospectral flows in loop algebras. This involves identifying certain reduced symplectic vector spaces through an equivariant moment map with a set of coadjoint orbits whose elements are rational functions of the complexified loop parameter. The flows induced by Hamiltonians from the ring of spectral invariants are determined, through the Adler-Kostant-Symes (AKS) theorem, by isospectral equations of Lax type,

$$\frac{d\mathcal{N}(\lambda)}{dt} = [(d\phi)_+, \mathcal{N}(\lambda)]\,, \tag{0.1}$$

* Research supported in part by the Natural Sciences and Engineering Research Council of Canada and by U.S. Army Grant DAA L03-87-K-0110