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Navier and Stokes Meet the Wavelet

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Abstract. We work in the space $\mathscr{F} = \mathscr{F}_\varepsilon$ of divergence-free measurable vector fields on R^3 complete in the norm $\| \ \|'$, where

$$(\| v \|')^2 = \operatorname*{Sup}_{\substack{x \\ R \leqq 1}} \left(\frac{1}{R}\right)^{1+\varepsilon} \int_{B(x,R)} v^2(y)\, d^3y$$

for some fixed $\varepsilon > 0$. $B(x, R)$ is the ball of radius R centered at x. Given an initial velocity distribution $\vec{v}(0)$ in $\mathscr{F}$, we find $\vec{v}(x, t)$ for $0 \leqq t \leqq T = T(\| v(0) \|')$, $T > 0$, such that $\vec{v}(x, t)$ is the unique strong solution of the Navier–Stokes equations, in a suitable sense.

We expand $\vec{v}'(x, t) = \vec{v}(x, t) - \vec{v}(x, 0)$ in terms of divergence-free vector wavelets $\{\vec{u}_\alpha\}$

$$\vec{v}'(x, t) = \sum_\alpha c_\alpha(t) \vec{u}_\alpha(x) .$$

The Navier–Stokes equations become an infinite set of integral equations for the $c_\alpha(t)$. In an appropriate space one realizes the $c_\alpha(t)$ satisfying the equations as the fixed point of a contraction mapping. The thus unique solution is the strong solution mentioned above.

Loosely Speaking. Given $\vec{v}(0)$ of finite $\| \ \|'$ norm, there is one and only one $\vec{v}(t)$ of bounded $\| \ \|'$ norm on $[0, T]$ with $T = T(\| v(0) \|') > 0$, that satisfies both

a) the Navier–Stokes equations

and

b) $\lim_{R \to \infty} \frac{1}{R^3} \int_{B(0,R)} [\vec{v}(x, t) - \vec{v}(x, 0)] = \vec{0}$, all t.

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