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Gravity in Non-Commutative Geometry

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Abstract. We study general relativity in the framework of non-commutative differential geometry. As a prerequisite we develop the basic notions of non-commutative Riemannian geometry, including analogues of Riemannian metric, curvature and scalar curvature. This enables us to introduce a generalized Einstein-Hilbert action for non-commutative Riemannian spaces. As an example we study a space-time which is the product of a four dimensional manifold by a two-point space, using the tools of non-commutative Riemannian geometry, and derive its generalized Einstein-Hilbert action. In the simplest situation, where the Riemannian metric is taken to be the same on the two copies of the manifold, one obtains a model of a scalar field coupled to Einstein gravity. This field is geometrically interpreted as describing the distance between the two points in the internal space.

1. Introduction

The poor understanding we have of physics at very short distances might lead one to expect that our description of space-time at tiny distances is inadequate. No convincing alternative description is known, but different routes to progress have been proposed. One such proposal is to try to formulate physics on some non-commutative space-time. There appear to be too many possibilities to do this, and it is difficult to see what the right choice is. So the strategy is to consider slight variations of commutative geometry, and to see whether reasonable models can be constructed. This is the approach followed by Connes [1], and Connes and Lott [2, 3]. They consider a model of commutative geometry (a Kaluza-Klein theory with an internal space consisting of two points), but use non-commutative geometry to define metric properties. The result is an economical way of deriving the standard model in which, roughly speaking, the Higgs field appears as the component of the gauge field in the internal direction.

In this paper, we show how gravity, in its simplest form, can be introduced in this context. We first propose a generalization of the basic notions of Riemannian

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