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The Topological Structure of the Unitary and Automorphism Groups of a Factor*

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Abstract. It is proved that a large class of II_1 factors have unitary group which is contractible in the strong operator topology, but whose fundamental group in the norm topology is isomorphic to the additive real numbers as proven by Araki-Smith-Smith [1]. The class includes the approximately finite dimensional factor of type II_1 and the group factor associated with the free group on infinitely many generators. This contractibility is used to prove the contractibility of the automorphism group of the approximately finite dimensional factor of type II_1 and type II_∞. It is further shown that the fundamental group of the automorphism group of the approximately finite dimensional factor of type III_λ, $0 < \lambda < 1$, is isomorphic to the integer group $\mathbb{Z}$.

Introduction

While the basic topological properties of the unitary group $U(n)$ of a finite dimensional Hilbert space, its homotopy groups etc., are well known for quite some time, the first results concerning the quantum theoretical setting of the infinite dimensional Hilbert space appeared only in the 60's and 70's. In 1965 Kuiper proved that in sharp contrast with the finite dimensional case the unitary group of an infinite dimensional Hilbert space, $U(\infty)$, endowed with the operator norm topology, is contractible [11]. This result was later extended to unitary groups $\mathcal{U}(\mathcal{M})$ of properly infinite von Neumann factors $\mathcal{M}$ by Breuer and Singer [4, 5]. However, for factors of type II_1 the situation is quite different, as H. Araki, L. Smith, and M.-S. B. Smith showed in 1971 that the first homotopy group $\pi_1(\mathcal{U}(\mathcal{M}))$ is then isomorphic to $\mathbb{R}$ and thus nontrivial, [1].

In recent years, the merge of topological methods in operator algebras and noncommutative geometry stimulated much interest in studying further topological properties of the unitary groups of more general operator algebras (with their operator norm topology).

In the case of von Neumann algebras though the natural topologies to consider are the weak and strong operator topologies, which for infinite dimensional $\mathcal{M}$ are

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