Commun. Math. Phys. 155, 71–92 (1993)

Communications in Mathematical Physics

The Analogues of Entropy and of Fisher's Information Measure in Free Probability Theory, I

Dan Voiculescu★

I.H.E.S., F-91440 Bures-sur-Yvette, France★★

Received June 9, 1992

Dedicated to Huzuhiro Araki

Abstract. Analogues of the entropy and Fisher information measure for random variables in the context of free probability theory are introduced. Monotonicity properties and an analogue of the Cramer-Rao inequality are proved.

Introduction

In [25] we began studying operator algebra free products from the probabilistic point of view. The idea is to look at free products as an analogue of tensor products and to develop a corresponding highly noncommutative probabilistic framework where freeness is given a treatment similar to independence. We showed [25] that there is a free central limit theorem with the semicircle law playing the role of the Gaussian distribution and that there is a functor from Hilbert spaces to operator algebras, which is the free analogue of the Gaussian functor of second quantization (i.e. the Gaussian process indexed by a Hilbert space). For the addition and multiplication of bounded free random variables we introduced corresponding free convolution operations on the distributions and constructed linearizing transforms, i.e. analogues of the logarithm of the Fourier and respectively Mellin transforms [26, 27]. In this context one-parameter free convolution semigroups correspond to one-dimensional quasilinear complex conservation laws satisfied by the Cauchy-transforms of the distributions, the complex Burger equation, in particular, being the analogue of the heat equation. This free harmonic analysis has been extended to distributions of unbounded random variables and the infinitely divisible laws have been studied [26, 4, 14, 5].

The explanation for the occurrence of the semicircle law, both in the free central limit theorem and in Wigner's work on asymptotics of large random matrices, was found in [29]. We showed that asymptotically, entrywise independence of large

★ Work supported in part by a Grant from the National Science Foundation

★★ Permanent address: Department of Mathematics, University of California, Berkeley, CA 94720, USA