Commun. Math. Phys. 155, 27–46 (1993)

Communications in Mathematical Physics

Optimal Hypercontractivity for Fermi Fields and Related Non-Commutative Integration Inequalities

Eric A. Carlen[1]* **and Elliott H. Lieb**[2]**

[1] Department of Mathematics, Princeton University, Princeton, New Jersey 08544, USA
[2] Departments of Mathematics and Physics, Princeton University, Princeton, New Jersey 08544–0708, USA

Received April 27, 1992; in revised form July 16, 1992

Dedicated to Huzihiro Araki

Abstract. Optimal hypercontractivity bounds for the fermion oscillator semigroup are obtained. These are the fermion analogs of the optimal hypercontractivity bounds for the boson oscillator semigroup obtained by Nelson. In the process, several results of independent interest in the theory of non-commutative integration are established.

I. Introduction

Observables pertaining to the configuration of a quantum system with n degrees of freedom are operators $Q_1, Q_2, \ldots, Q_n$ which, depending on the system, may or may not commute. Our main concern is with the case in which the configuration variables are amplitudes of certain field modes.

For boson fields, these configuration observables do commute, and the state space $\mathscr{H}$ can be taken as the space of all complex square integrable functions on their joint spectrum. This is the Schrödinger q-space representation, and the fact that in it the state space is a function space, and not just an abstract Hilbert space, is very helpful in the analysis of such systems. As one example, it sometimes turns out that physically interesting operators preserve the cone of positive functions, and this opens the way to the application of the Perron-Frobenius theorem in the study of ground states of such systems.

For fermion fields, the configuration observables do not commute, and this simple q-space representation is not available. However, the non-commutative integration theory of Irving Segal [Se53] permits the construction of a suitable substitute, and in fact it was created with such a purpose in view. This approach to the study of fermion systems has been extensively developed by Gross [Gr72] who, among other things, proved a version of the Perron-Frobenius theorem adapted to

* On leave from School of Math., Georgia Institute of Technology, Atlanta, GA 30332, USA
** Work supported by U.S. National Science Foundation grant no. PHY90–19433–A01.