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Real Killing Spinors and Holonomy

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Abstract. We give a description of all complete simply connected Riemannian manifolds carrying real Killing spinors. Furthermore, we present a construction method for manifolds with the exceptional holonomy groups G_2 and Spin(7).

1. Introduction

Let M be an n-dimensional complete Riemannian spin manifold. A spinor field ψ is called *Killing spinor* with Killing constant α if for all tangent vectors X the equation $\nabla_X \psi = \alpha \cdot X \cdot \psi$ holds. Here $X \cdot \psi$ denotes the Clifford product of X and ψ. Killing spinors occur in physics, e.g. in supergravity theories, see [11], but they are also of mathematical interest. Friedrich showed that if M is compact and the scalar curvature satisfies $S \geqq S_0 > 0$, $S_0 \in \mathbb{R}$, then for all eigenvalues λ of the Dirac operator the estimate $\lambda^2 \geqq \frac{1}{4}\frac{n}{n-1} S_0$ holds, see [13]. If we have equality in this estimate, then the corresponding eigenspinor is a Killing spinor.

If M carries a Killing spinor, then M is an Einstein manifold with Ricci curvature $\mathrm{Ric} = 4(n-1)\alpha^2$. In particular, we have three distinct cases; α can be purely imaginary, then M is noncompact and we call ψ an imaginary Killing spinor, α can be 0, in this case ψ is a parallel spinor field, and finally α can be real, then M is compact and ψ is called a real Killing spinor. This terminology is somewhat misleading, because a real Killing spinor is not necessarily a real spinor field; we *always* work with complex spinor fields.

Hitchin showed that manifolds with parallel spinor fields can be characterized by their holonomy group, see [28, Th. 1.2 and footnote p. 54]. See also [15] and [35].

Manifolds with imaginary Killing spinors have been classified by Baum in [1–3], shortly later the classification has been extended by Rademacher to generalized imaginary Killing spinors where we allow the Killing "constant" α to be an imaginary function, see [32].

Most results on real Killing spinors known so far are statements for particular (low) dimensions. For example, Friedrich showed in [14] that a complete