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Distribution of the Error Term for the Number of Lattice Points Inside a Shifted Circle

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Abstract. We investigate the fluctuations in $N_\alpha(R)$, the number of lattice points $n \in \mathbf{Z}^2$ inside a circle of radius R centered at a fixed point $\alpha \in [0, 1)^2$. Assuming that R is smoothly (e.g., uniformly) distributed on a segment $0 \leqq R \leqq T$, we prove that the random variable $\frac{N_\alpha(R) - \pi R^2}{\sqrt{R}}$ has a limit distribution as $T \to \infty$ (independent of the distribution of R), which is absolutely continuous with respect to Lebesgue measure. The density $p_\alpha(x)$ is an entire function of x which decays, for real x, faster than $\exp(-|x|^{4-\varepsilon})$. We also obtain a lower bound on the distribution function $P_\alpha(x) = \int_{-\infty}^x p_\alpha(y)\,dy$ which shows that $P_\alpha(-x)$ and $1 - P_\alpha(x)$ decay when $x \to \infty$ not faster than $\exp(-x^{4+\varepsilon})$. Numerical studies show that the profile of the density $p_\alpha(x)$ can be very different for different α. For instance, it can be both unimodal and bimodal. We show that $\int_{-\infty}^{\infty} x p_\alpha(x)\,dx = 0$, and the variance $D_\alpha = \int_{-\infty}^{\infty} x^2 p_\alpha(x)\,dx$ depends continuously on α. However, the partial derivatives of D_α are infinite at every rational point $\alpha \in \mathbf{Q}^2$, so D_α is analytic nowhere.

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