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The Krichever Map, Vector Bundles Over Algebraic Curves, and Heisenberg Algebras*

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Abstract. We study the Grassmannian Gr_X^n consisting of equivalence classes of rank n algebraic vector bundles over a Riemann surface X with an holomorphic trivialization at a fixed point p. Commutative subalgebras of $gl(n, H_\lambda)$, H_λ being the ring of functions holomorphic on a punctured disc about p, define flows on the Grassmannian, giving rise to classes of solutions to multi-component KP hierarchies. These commutative subalgebras correspond to Heisenberg algebras in the Kac–Moody algebra associated to $gl(n, H_\lambda)$. One can obtain, by the Krichever map, points of Gr_X^n (and solutions of mcKP) from coverings $f\colon Y \to X$ and other geometric data. Conversely for every point of Gr_X^n and for every choice of Heisenberg algebra we construct, using the cotangent bundle of Gr_X^n, an algebraic curve covering X and other data, thus inverting the Krichever map. We show the explicit relation between the choice of Heisenberg algebra and the geometry of the covering space.

1. Introduction

1.1. In the seventies it was discovered that one could obtain solutions of certain non-linear evolutionary equations of "soliton type" in terms of θ-functions of Riemann surfaces or, more generally, one could construct solutions from coherent sheaves on algebraic curves, see e.g., [Kr, Mum1, Du, KrN, vMM]. (In fact, certain solutions of the Korteweg–de Vries equation (KdV) in terms of elliptic functions were known classically [KdV].) Somewhat later it was discovered that solutions of such equations could be identified with points of infinite dimensional Grassmannians, or, equivalently, with the orbits of infinite dimensional groups in representation spaces (see e.g., [Sa, DaJKM, Ka]).

Both points of view, the algebro-geometric one and the representation theoretic, could be connected by associating to a Riemann surface (along with a line

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