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A Dynamical Approach to Symplectic and Spectral Invariants for Billiards

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Abstract. The Lazutkin parameter for curves which are invariant under the billiard ball map is viewed symplectically in a way which makes it analogous to the sum of the values of a generating function over a closed orbit. This leads to relations among lengths of closed geodesics, lengths of invariant curves for the billiard map, rotation numbers, and the Lazutkin parameter. These relations establish the Birkhoff invariant and the expansion for the lengths of invariant curves in terms of the Lazutkin parameter as symplectic and spectral invariants (for the Dirichlet spectrum) and provide invariants which characterize a family of ellipses among smooth curves with positive curvature.

Geodesic flow on a bounded planar region gives rise to several geometric objects among which are closed reflected geodesics and invariant curves – closed curves whose tangents are invariant under reflection at the boundary. On a bounded domain, the map that assigns to each geodesic segment its successor after reflection at the boundary is called the billiard ball map and its dual (in the cotangent bundle for the boundary) is called the boundary map.

As shown by Guillemin and Melrose in [9], the lengths of closed geodesics are symplectic invariants associated with a generating function for the boundary map. Moreover, in a convex planar domain Marvizi and Melrose defined the wave invariants (see [12]), obtained via the interpolating hamiltonian of [14] which takes account of the singularity of the boundary map at the boundary. These wave invariants are symplectic and spectral invariants when viewed as functions of the rotation number associated to closed geodesics, but they do not have, as a whole, direct dynamical or geometric interpretations.

In this paper a new set of invariants, the caustics' invariants, are introduced. They are the lengths of invariant curves viewed as functions of the Lazutkin parameter of [11]. The Lazutkin parameter has a dynamical interpretation obtained by modifying a generating function for the boundary map.

Using these symplectic invariants one can isolate ellipses among planar domains whose curvature is strictly positive. In Sect. 6 it is shown that the caustics' invariants can be obtained from the wave invariants and the lengths of closed geodesics, and