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The Landau–Lifshitz Equation, Elliptic Curves and the Ward Transform

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Abstract. The Landau–Lifshitz (LL) equation is studied from a point of view that is close to that of Segal and Wilson's work on KdV. The LL hierarchy is defined and shown to exist using a dressing transformation that involves parameters $\lambda_1, \lambda_2, \lambda_3$ that live on an elliptic curve Σ. The crucial role of the group $K \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ of translations by the half-periods of Σ and its non-trivial central extension $\tilde{K}$ is brought out and an analogue of Birkhoff factorisation for $\tilde{K}$-equivariant loops in Σ is given. This factorisation theorem is given two treatments, one in terms of the geometry of an infinite-dimensional Grassmannian, and the other in terms of the algebraic geometry of bundles over Σ. Further, a Ward-like transform between a class of holomorphic vector bundles on the total space Z of a line-bundle over Σ and solutions of LL is constructed. An appendix is devoted to a careful definition of the Grassmannian of the Frechet space $C^\infty(S^1)$.

1. Introduction

This paper aims to expose the links between a variety of methods for solving completely integrable non-linear equations in the special case of the Landau–Lifshitz (LL) equation (cf. (2.23)). The interest in this example stems from the fact that the spectral curve Σ which arises from the Lax form of the equations is an elliptic curve. This means [13] that there is no immediate generalisation of the methods which are applicable when the spectral curve is the Riemann sphere. For example, a generic $\mathrm{SL}_2(\mathbb{C})$-valued loop on the unit circle has a Birkhoff factorisation as a product of loops, one holomorphic inside the unit disc, the other holomorphic outside the disc. There is however no such factorisation in general for a disc in Σ. It emerges that the only loops that arise in the study of the LL equation behave in a prescribed fashion under a discrete group of symmetries of Σ and for such loops an appropriate analogue of Birkhoff factorisation *does* exist (cf. Theorems 3.1 and 4.1 below).

Previous literature on the LL equation may be traced from [7]. Our motivation for its study stemmed partly from the relation to conformal field theory that is