Commun. Math. Phys. 153, 559–577 (1993)

Communications in Mathematical Physics

Methods of KAM-Theory for Long-Range Quasi-Periodic Operators on $\mathbb{Z}^\nu$. Pure Point Spectrum

V. A. Chulaevsky[1] **and E. I. Dinaburg**[2]

[1] Department of Mathematics, University of California at Irvine, Irvine, CA 92717, USA. Permanent address: Institute of Mathematical Problems in Biology, Russian Academy of Sciences, Pushchino, Moscow Region 142292, Russia
[2] O. Yu. Schmidt Institute of Earth Physics, Russian Academy of Sciences, B. Gruzinskaya ul., 10, Moscow 123810, Russia

Received July 20, 1991; in revised form September 10, 1992

Abstract. We consider the class of quasi-periodic self-adjoint operators $\hat{H}(x) = \hat{D}(x) + \hat{V}(x)$, $x \in S^1 = \mathbb{R}^1/\mathbb{Z}^1$, on a multi-dimensional lattice $\mathbb{Z}^\nu$, with the matrix elements

$$\hat{D}_{mn}(x) = \delta_{mn} D(x + n\omega)\,, \qquad \hat{V}_{mn}(x) = V(m-n, x+n\omega)\,,$$

where $D(x+1) = D(x)$, $V(n, x+1) = V(n,x)$, $\omega \in \mathbb{R}^\nu$, and $|V(n,x)| \le \varepsilon e^{-r|n|}$, $r > 0$. We prove that, if ε is small enough, $V(n,\cdot)$ and $D(\cdot)$ satisfy some conditions of smoothness, and $D(\cdot)$ is non-degenerate, then for a.e. ω and for a.e. $x \in S^1$ the operator $\hat{H}(x)$ has pure point spectrum. All its eigenfunctions belong to $l^1(\mathbb{Z}^\nu)$.

1. Introduction

In the spectral theory of almost periodic media, two important classes of quantum Hamiltonians have been investigated particularly well: nearest-neighbor Hamiltonians like the "almost-Mathieu" operator on $\mathbb{Z}^1$,

$$(H_\varepsilon(x)\psi)(n) = \varepsilon(\psi(n-1) + \psi(n+1)) + \cos(x + n\omega)\psi(n)\,,$$

which describes a quasi-periodic medium with infinite number of resonances (Sinai [1], Fröhlich, Spencer, and Wittwer [2]), and long-range Hamiltonians like

$$(H_\varepsilon(x)\psi)(n) = \varepsilon \sum_{n\in\mathbb{Z}^\nu} a(n-m)\psi(m) + \tan(x+n\omega)\psi(n)\,,$$

with $|a(n)| \le e^{-r|n|}$, $r > 0$, which describe media with no resonances (see Bellissard, Lima, and Scoppola [3]). The main purpose of the present paper is to extend the perturbation-theoretic analysis of resonances, originally proposed by Sinai [1] and going back to the KAM (Kolmogorov-Arnold-Moser) theory. Many authors mentioned that the methods of the KAM theory appear naturally in localization problems (see in particular [4]). We refer also to a related work by Bellissard [6].