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On the Long Time Behavior of the Doubly Infinite Toda Lattice under Initial Data Decaying at Infinity

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Abstract. We provide rigorous analysis of the long time behavior of the (doubly infinite) Toda lattice under initial data that decay at infinity, in the absence of solitons. We solve (approximately and for large times) the Riemann-Hilbert matrix factorization problem equivalent to the related inverse scattering problem with the help of the Beals-Coifman formula, by reducing it to a simpler one through a series of contour deformations in the spirit of the Deift-Zhou method.

0. Introduction

0.1. Statement of the Problem and Results

In this paper we consider the doubly infinite Toda lattice under initial conditions decaying at infinite. More precisely, we provide the full analysis of the long-time behavior of the initial value problem:

$$\begin{aligned} \dot{a}_n &= 2(b_n^2 - b_{n-1}^2)\,, \\ \dot{b}_n &= b_n(a_{n+1} - a_n)\,, \end{aligned} \qquad \infty < n < \infty \tag{0.1}$$

with initial conditions:

$$a_n(0) = a_n^0\,, \qquad b_n(0) = b_n^0\,, \tag{0.2}$$

such that a_n^0 and $b_n^0 - 1/2$ decay faster that polynomially at ∞ and $-\infty$.

We will assume here that there are no solitons. In other words the initial data are such that the underlying linear discrete Lax operator has no eigenvalues. The solitons will be added in a later study.

It turns out that for large times t one can distinguish three different regions.

1. In the region $|n/t| < 1 - C/t^{2-\delta}$, where C, δ are any given positive constants, the lattice performs decaying oscillations of order $0(1/t^{1/2})$. More precisely,

$$4b_n^2 - 1 = s_n - s_{n-1}\,, \tag{0.3}$$