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Linear Adiabatic Theory. Exponential Estimates

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Abstract. A general adiabatic expansion is written down. The basic result is that, under appropriate smoothness conditions, the adiabatic estimations can be pushed up to exponential order. The results imply exponential estimates not only for hamiltonians analytic in a neighbourhood of the real axis but also for hamiltonians which (in an appropriate sense) beong to Gevrey classes.

1. The Problem, Heuristics, Results

For definiteness in this section we shall consider unitary evolutions in Hilbert spaces. However, the main results in Sect. 2 hold true in a more general context.

Consider the evolution, $U_\varepsilon(s, s_0)$, given by

$$i\varepsilon \frac{d}{ds} U_\varepsilon(s, s_0) = H(s) U_\varepsilon(s, s_0); \qquad U_\varepsilon(s_0, s_0) = 1 \tag{1.1}$$

in the limit $\varepsilon \longrightarrow 0$. Since, as it is well known, (1.1) is hard to integrate, one can pose the problem to obtain information about $U_\varepsilon(s, s_0)$ without actually integrating (1.1). Our strategy is to proceed in two steps:

I) Find out (almost) invariant subspaces, $\mathscr{K}(s;\varepsilon)$, under the evolution $U_\varepsilon(s, s_0)$, i.e.

$$U_\varepsilon(s, s_0)\mathscr{K}(s_0;\varepsilon) \cong \mathscr{K}(s;\varepsilon)\,. \tag{1.2}$$

II) Integrate the evolution equation restricted to the subspaces $\mathscr{K}(s;\varepsilon)$. Let us mention that this step is trivial if (as it is the case in many physical applications) $\dim \mathscr{K}(s;\varepsilon) = 1$.

In the rest of this section we shall describe, at the heuristic level, our procedure to solve I) and II) and outline the relation of our results with some of the previously

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