Commun. Math. Phys. 152, 433–478 (1993)

Communications in
Mathematical
Physics

The Cauchy Problem for Non-Linear Klein-Gordon Equations

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Received February 25, 1992; in revised form July 23, 1992

Abstract. We consider in $\mathbb{R}^{n+1}$, $n \geqq 2$, the non-linear Klein-Gordon equation. We prove for such an equation that there is a neighbourhood of zero in a Hilbert space of initial conditions for which the Cauchy problem has global solutions and on which there is asymptotic completeness. The inverse of the wave operator linearizes the non-linear equation. If, moreover, the equation is manifestly Poincaré covariant then the non-linear representation of the Poincaré Lie algebra, associated with the non-linear Klein-Gordon equation is integrated to a non-linear representation of the Poincaré group on an invariant neighbourhood of zero in the Hilbert space. This representation is linearized by the inverse of the wave operator. The Hilbert space is, in both cases, the closure of the space of the differentiable vectors for the linear representation of the Poincaré group, associated with the Klein-Gordon equation, with respect to a norm defined by the representation of the enveloping algebra.

1. Introduction

The problem of the existence of global solutions for the non-linear Klein-Gordon equation

$$(\Box + m^2)\varphi(t,x) = P\left(\varphi(t,x), \frac{\partial}{\partial t}\varphi(t,x), \nabla\varphi(t,x)\right), \qquad m^2 > 0, \tag{1.1}$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\varphi(t,x) \in \mathbb{C}$, $\nabla = (\partial_1, \dots, \partial_n)$, $\partial_i = \dfrac{\partial}{\partial x_i}$, $\Delta = \sum_{i=1}^{n} \partial_i^2$, $\Box = \dfrac{\partial^2}{\partial t^2} - \Delta$, and

$n \geqq 1$, has been studied by various authors during the last two decades under different hypotheses on P and n. It is difficult to give here an exhaustive description of the results already obtained and we shall only mention some of the results which, we believe, are the most significant for the case where P is a C^∞ function vanishing at zero together with its first derivatives.