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Classical A_n-W-Geometry

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Abstract. By analyzing the *extrinsic* geometry of two dimensional surfaces chirally embedded in $\mathscr{C}P^n$ (the $\mathscr{C}P^n$ W-surface [1]), we give exact treatments in various aspects of the classical W-geometry in the conformal gauge: First, the basis of tangent and normal vectors are defined at regular points of the surface, such that their infinitesimal displacements are given by connections which coincide with the vector potentials of the (conformal) A_n-Toda Lax pair. Since the latter is known to be intrinsically related with the W symmetries, this gives the geometrical meaning of the A_n W-Algebra. Second, W-surfaces are put in one-to-one correspondence with solutions of the conformally-reduced WZNW model, which is such that the Toda fields give the Cartan part in the Gauss decomposition of its solutions. Third, the additional variables of the Toda hierarchy are used as coordinates of $\mathscr{C}P^n$. This allows us to show that W-transformations may be extended as particular diffeomorphisms of this target-space. Higher-dimensional generalizations of the WZNW equations are derived and related with the Zakharov-Shabat equations of the Toda hierarchy. Fourth, singular points are studied from a global viewpoint, using our earlier observation [1] that W-surfaces may be regarded as instantons. The global indices of the W-geometry, which are written in terms of the Toda fields, are shown to be the instanton numbers for associated mappings of W-surfaces into the Grassmannians. The relation with the singularities of W-surface is derived by combining the Toda equations with the Gauss-Bonnet theorem.

1. Introduction

The geometry behind W-algebras [2, 3] has been an important open problem of solvable quantum field theories ([4–13]). It is supposed to give generalizations of the

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