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A Center-Stable Manifold Theorem for Differential Equations in Banach Spaces

Th. Gallay

Département de Physique Théorique, Université de Genève, CH-1211 Geneva 4, Switzerland

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Abstract. We prove a center-stable manifold theorem for a class of differential equations in (infinite-dimensional) Banach spaces.

1. Introduction

The center-stable manifold theorem is a standard tool in analyzing the behavior of a differentiable dynamical system in the vicinity of a stationary point. In its usual formulation, this theorem applies to smooth maps or flows in (finite or infinite-dimensional) Banach spaces (see for example Ruelle [1]). This framework is, however, too restrictive for many interesting applications, especially in the realm of partial differential equations. Indeed, even in the simple example of the heat equation $\partial_t u = \Delta u$, the solution curves do not define a flow in the function space, but only a semiflow, and no general theorem seems to be available in such cases. Moreover, in some elliptic differential problems where the spectrum of the linear operator is unbounded in the unstable direction, it is not even possible to associate a semiflow with the equation, since arbitrarily small initial data may diverge in arbitrarily short times. Nevertheless, center manifold techniques have been successfully applied to such problems, see Mielke [2].

It is thus important to formulate a center-stable manifold theorem directly for the differential equation itself, with no reference to any flow possibly associated with it. In this paper, we prove such a theorem for a class of equations characterized by weak assumptions on the linear operator in the right-hand side, but imposing relatively restrictive conditions on the nonlinear terms (smoothness). The main motivation for this paper was to provide the mathematical apparatus for the companion paper, written jointly with J.-P. Eckmann [9], where the results presented here are applied to the problem of constructing front solutions for the Ginzburg–Landau equation with complex amplitudes. Our approach follows closely Eckmann and Wayne [3], but provides additional information on the regularity in time of the solutions. Analogous results for more general nonlinearities and for non-autonomous equations can be found in Mielke [2, 4, 5]. For