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Front Solutions for the Ginzburg–Landau Equation

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Abstract. We prove the existence of front solutions for the Ginzburg–Landau equation

$$\partial_t u(x, t) = \partial_x^2 u(x, t) + (1 - |u(x, t)|^2)u(x, t) ,$$

interpolating between two stationary solutions of the form $u(x) = \sqrt{1 - q^2}e^{iqx}$ with different values of q at $x = \pm\infty$. Such fronts are shown to exist when at least one of the q is in the Eckhaus-unstable domain.

1. Introduction

We consider the Ginzburg–Landau equation (GL)

$$\partial_t u(x, t) = \partial_x^2 u(x, t) + (1 - |u(x, t)|^2)u(x, t) , \tag{1.1}$$

where u is a complex-valued function of $x \in \mathbf{R}$ and $t \in \mathbf{R}_+$. This equation has time-independent periodic solutions of the form

$$u_q(x) = \sqrt{1 - q^2}e^{i\varphi}e^{iqx} , \tag{1.2}$$

where $q \in [-1, 1]$ and $\varphi \in \mathbf{R}$. These stationary solutions are known to be unstable for small amplitudes ($q^2 > 1/3$) and marginally stable for large amplitudes ($q^2 < 1/3$) (*Eckhaus stability*, cf. [CE]).

Our aim is to show the existence of *front solutions* of Eq. (1.1) interpolating between two stationary solutions (1.2). By this, we mean solutions of the form $u(x, t) = U(x, x - ct)$, where $U(x, \xi)$ is a complex function which converges to one of the stationary solutions (1.2), say $u_{q_0}(x)$, as $\xi \to -\infty$ and to another one, say $u_{q_1}(x)$, as $\xi \to +\infty$. Such solutions typically look like a fixed envelope moving to the right with constant velocity $c > 0$, while leaving a periodic pattern (the function u_{q_0}) behind and destroying another one (u_{q_1}) in front, as shown in Fig. 1.

In the case where $u_{q_1} \equiv 0$ ($q_1 = \pm 1$), solutions of this form are easily shown to exist, see e.g., [CE, B]. Indeed, inserting in Eq. (1.1) the ansatz $u(x, t)$