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A Combinatorial Approach to Topological Quantum Field Theories and Invariants of Graphs*

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Dedicated to H. Araki and E. Lieb on the occasion of their 60th birthdays

Abstract. The combinatorial state sum of Turaev and Viro for a compact 3-manifold in terms of quantum $6j$-symbols is generalized by introducing observables in the form of coloured graphs. They satisfy braiding relations and allow for surgeries and a discussion of cobordism theory. Application of these techniques give the dimension and an explicit basis for the vector space of the topological quantum field theory associated to any Riemann surface with arbitrary coloured punctures.

1. Introduction

Since the early days of topological quantum field theories there was the question whether such field theories have a lattice formulation analogous to lattice gauge theory. The reason is that one would like to work in a context with mathematically well defined quantities instead of more or less formal functional integrals. This question has been answered affirmatively in part by the work of Turaev and Viro [TV]. Invoking the $6j$-symbols for the quantum group $U_q(sl(2, \mathbf{C})$ with q being a $2r^{\text{th}}$ primitive root of unity they constructed invariants $Z(M)$ of closed, compact 3-manifolds M. In [KMS] this construction was extended to compact 3-manifolds with boundary. For orientable 3-manifolds, the case we shall exclusively be dealing with in this article, these invariants, called state sums or partition functions, in the case $\partial M = \emptyset$ satisfy the relation

$$Z(M) = \tau(M)\tau(M^*) = |\tau(M)|^2, \tag{1.1}$$

where $\tau(M)$ is the partition function for the $SU(2)$-Chern Simons topological quantum field theory at level $k = r - 2$ [T1]. Originally $\tau(M)$ was introduced and discussed on a formal level based on functional integration methods [Wi2]. However, introducing the theory of coloured ribbon graphs, Reshetikhin and Turaev [RT] have provided a mathematical construction of $\tau(M)$ having all the desired properties. Now the

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