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Polynomial Averages in the Kontsevich Model

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Abstract. We obtain in closed form averages of polynomials, taken over hermitian matrices with the Gaussian measure involved in the Kontsevich integral, and prove a conjecture of Witten enabling one to express analogous averages with the full (cubic potential) measure, as derivatives of the partition function with respect to traces of inverse odd powers of the external argument. The proofs are based on elementary algebraic identities involving a new set of invariant polynomials of the linear group, closely related to the general Schur functions.

1. Introduction

In their papers on intersection theory on moduli spaces of Riemann surfaces, Witten [1] and Kontsevich [2] discussed certain identities on matrix integrals. We provide here algebraic proofs for these statements which read as follows.

Let $X, Y, \Lambda, \ldots$ denote $N \times N$ hermitian matrices. For Λ positive definite, hence Λ^{-1} well defined, introduce the measure

$$\begin{aligned} d\mu_\Lambda^{(N)}(Y) &= (2\pi)^{-N^2/2} \prod_{i=1}^{N} dY_{ii} \prod_{1\le i<j\le N} d\,\mathrm{Re}\, Y_{ij}\, d\,\mathrm{Im}\, Y_{ij} \exp -\frac{1}{2}\mathrm{tr}\Lambda Y^2 \\ &= dY \exp -\frac{1}{2}\mathrm{tr}\Lambda Y^2 \,. \end{aligned} \tag{1.1}$$

Proposition (K). *For any polynomial P in the traces of odd powers of Y ("odd traces" for short), there exists a polynomial Q in odd traces of Λ^{-1}, such that, independently of N large enough*

$$\langle P\rangle_{(N)}(\Lambda^{-1}) = \frac{\int d\mu_\Lambda^{(N)}(Y) P(Y)}{\int d\mu_\Lambda^{(N)}(Y)} = Q(\Lambda^{-1})\,. \tag{1.2}$$

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